J. Fluid Mech. (1995), *vol.* 284, *pp.* 137–158

On the stability of large-amplitude geostrophic flows in a two-layer fluid: the case of 'strong' beta-effect

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(Received 18 October 1993 and in revised form 25 August 1994)

This paper examines the stability of two-layer geostrophic flows with large displacement of the interface and strong β-effect. Attention is focused on flows with non-monotonic interface profiles which are not covered by the Rayleigh-style stability theorems proved by Benilov (1992*a*, *b*) and Benilov & Cushman-Roisin (1994). For such flows the coefficient of the highest derivative in the corresponding boundary-value problem vanishes at the point where the depth profile has an extremum. Although this singularity is similar to a critical level, it cannot be regularized by the simplistic introduction of infinitesimal viscosity through the assumption that the phase speed of the disturbance is complex. In order to regularize the singularity properly, one should consider the problem within the framework of the original ageostrophic viscous equations and, having obtained the boundary-value problem for harmonic disturbance, take the limit Rossby number $\rightarrow 0$, viscosity $\rightarrow 0$.

The results obtained analytically and (for special cases) numerically indicate that the stability of flows with non-monotonic profiles strongly depends on the depth of the upper layer. If the upper layer is 'thick' (i.e. if the average depth H_1 of the upper layer is of the order of the total depth of the fluid H_0), the stability boundary-value problem does not have any solutions at all, which means stability (however, this stability is structurally unstable, and the flow, generally speaking, can be made weakly unstable by any small effect such as external forcing, viscosity, or ageostrophic corrections). In the case of 'thin' upper layer ($H_1/H_0 \lesssim Ro$), the order of the singularity changes and all non-monotonic flows are unstable regardless of their profiles. It is also demonstrated that thin-upper-layer flows do not have to be non-monotonic to be unstable: if $u-\beta R_0^2$ (where u is the zonal velocity, β is the β-parameter, and R_0 is the deformation radius) changes sign somewhere in the flow, the stability boundary-value problem has another singular point which leads to instability.

1. Introduction

Consider a two-layer flow between two rigid planes (see figure 1). Assuming that the displacement of the interface is of the order of the depth of the upper layer, we shall introduce three governing parameters:

(i) the Rossby number

$$Ro = U/(fL),$$

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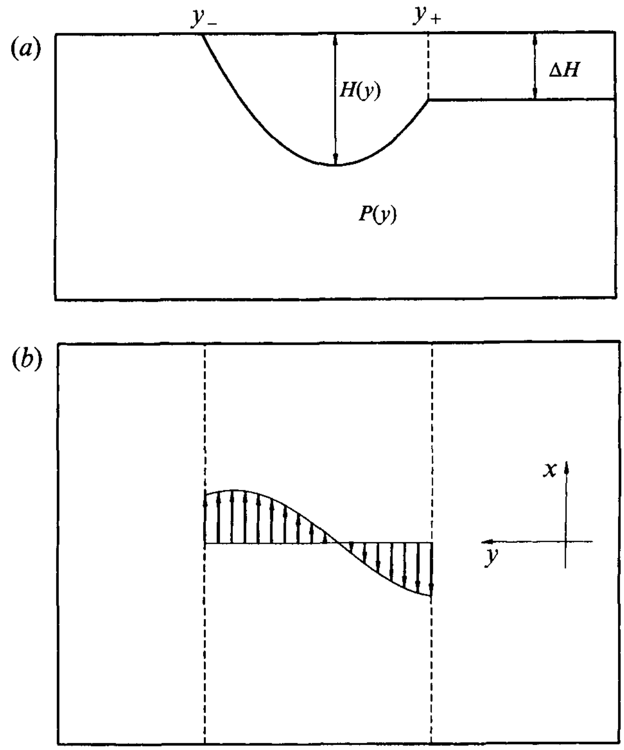

FIGURE 1. Formulation of the problem. (a) Crossection of the flow. (b) Bird's eye view.

where U is the effective velocity scale, L is the horizontal spatial scale of the motion and f is the Coriolis parameter;

(ii) the β-effect number

$$\alpha = R_0 \cot\theta / R_e,$$

where

$$R_0 = (g' H_0)^{1/2}/f, \tag{1.1}$$

$g' = g\,\delta\rho/\rho_0$ is the reduced acceleration due to gravity, ρ_0 and $(\rho_0 + \delta\rho)$ are the densities of the layers, H_0 is the total depth of the fluid, θ is the latitude, R_e is the earth's radius;

(iii) relative depth of the upper layer

$$\delta = H_1/H_0,$$

where H_1 is the depth scale of the upper layer.

The question of which range of Ro, α and δ is relevant to the real ocean was addressed by Benilov & Reznik (1994). Using Roden's (1975) and Nowlin & Klinck's (1986) experimental data on frontal currents in the Pacific and Southern oceans, they demonstrated that the most 'populated' regime is

$$Ro \ll 1, \quad \alpha \sim Ro.$$

See table 1. It is worth noting that all but one of the flows in table 1 have *thin* upper layer: $\delta \sim Ro$. We shall also note that the other two jets of the Antarctic Circumpolar Current (not mentioned in table 1) correspond to the weak β-effect.

Our attention is focused on the stability of zonal flows, where the depth of the upper layer H depends only on the meridional variable y (the non-dimensional variables H and y are scaled by H_0 and R_0, respectively). The stability properties of flows with

	Ro	α	δ
Kuroshio	0.040	0.021	0.109
Oyashio	0.026	0.012	0.073
Subarctic front	0.021	0.015	0.091
Subtropical front	0.050	0.044	0.091
Middle jet of ACC	0.011	0.004	0.400

TABLE 1. Ranges of Rossby number, β-effect number and relative depth of the upper layer in real oceans

'*thick*' upper layer ($\delta \sim 1$) were studied by Benilov (1992*a*), who demonstrated that all flows with monotonic profile are stable†:

$$H_y(y) \text{ does not change sign} \Rightarrow \text{stability.} \tag{1.2}$$

Benilov & Cushman-Roisin (1994) derived a similar stability condition:

$$H_y \text{ and } H_y + \alpha \text{ do not change sign} \Rightarrow \text{stability}$$

for flows with '*thin*' upper layer ($\delta \sim Ro$). Finally, condition (1.2) was found to also guarantee the stability of flows with '*very thin*' upper layer (Benilov 1992*b*).

It should be emphasized, however, that the above-mentioned stability conditions are only *sufficient* criteria; therefore flows, which do not satisfy them, are not necessarily unstable, and their stability properties are unclear. At the same time, oceanographic data (e.g. Roden 1975) suggest that there are a number of flows with non-monotonic profiles in the real ocean, where the velocity in the upper layer changes direction and $H_y = 0$ (e.g. the Oyashio current or subtropical front). The other conditions of stability, $H_y \neq -\alpha$, can be violated by any sufficiently strong eastward current (e.g. the subarctic frontal flow).

This paper examines the stability of zonal flows with $H_y = 0$ or $H_y = -\alpha$. The regimes of 'very thin' ($\delta \sim Ro^2$), 'thin' ($\delta \sim Ro$), and 'thick' ($\delta \sim 1$) upper layer will be considered in §§2, 3 and 4, respectively.

2. Stability of flows with 'very thin' upper layer: $\delta \sim Ro^2$

2.1. *Governing equations*

Consider a two-layer flow between two rigid planes (see figure 1). We shall introduce the following non-dimensional spatial variables and time:

$$x = \tilde{x}/R_0, \quad y = \tilde{y}/R_0, \quad t = \tilde{t}f;$$

where the dimensional variables are marked with tildes, f is the Coriolis parameter and R_0 is given by (1.1). Within the framework of the rigid-lid approximation the depths of the layers can be expressed in terms of a single non-dimensional variable h:

$$\tilde{h}_1 = H_0 h(x,y,t), \quad \tilde{h}_2 = H_0[1-h(x,y,t)];$$

where H_0 is the total depth of the fluid. We shall also introduce the non-dimensional velocity of the fluid in the upper layer:

$$(u,v) = \frac{(\tilde{u},\tilde{v})}{fR_d}.$$

† Benilov (1992*a*) claimed to have proved the stability of *all* flows for this case; however, he did not notice the divergence of the integrals in his stability theorem at the points where $H_y = 0$ (if any).

If the depth of the upper layer is much thinner than H_0, the flow induced in the lower layer is weak and can be neglected. As a result, we can use the so-called one-layer reduced-gravity model:

$$\left.\begin{aligned} u_t+uu_x+vu_y+h_x &= (1+\alpha y)\,v-\nu u+F^{(x)}, \\ v_t+uv_x+vv_y+h_y &= -(1+\alpha y)\,u-\nu v+F^{(y)}, \\ h_t+(uh)_x+(vh)_y &= 0; \end{aligned}\right\} \tag{2.1}$$

where

$$\nu = \tilde{\nu}/f$$

is the coefficient of bottom friction and $(F^{(x)}, F^{(y)})$ is the external forcing. We shall assume that the layer is thin:

$$h = \epsilon^2 h', \tag{2.2a}$$

where the new variable h' is of the order of unity and ϵ is a small parameter. The motion is assumed weak

$$u = \epsilon^2 u', \quad v = \epsilon^2 v', \tag{2.2b}$$

and slow

$$t = \epsilon^{-2} t'. \tag{2.2c}$$

In order to identify ϵ with the Rossby number, we should 'compress' the horizontal variables:

$$x = \epsilon x', \quad y = \epsilon y'. \tag{2.2d}$$

We shall also assume that the β-effect is weak:

$$\alpha = \epsilon^2 \alpha'. \tag{2.2e}$$

We shall not dwell on equalities (2.2*a–e*), but refer to Cushman-Roisin (1986) where the parameter space of system (2.1) is discussed in detail. We shall consider the regime where the viscosity and forcing are of the order of, or smaller than, the ageostrophic terms:

$$\nu = \epsilon\nu', \quad F^{(x)} = \epsilon F^{(x)'}, \quad F^{(y)} = \epsilon F^{(y)'}. \tag{2.2f}$$

Substitution of (2.2) into (2.1) yields (primes omitted):

$$\left.\begin{aligned} \epsilon^2 u_t+\epsilon(uu_x+vu_y)+h_x &= (1+\epsilon\alpha y)\,v+\epsilon(-\nu u+F^{(x)}), \\ \epsilon^2 v_t+\epsilon(uv_x+vv_y)+h_y &= -(1+\epsilon\alpha y)\,u+\epsilon(-\nu v+F^{(y)}), \end{aligned}\right\} \tag{2.3a}$$

$$\epsilon h_t+(uh)_x+(vh)_y = 0. \tag{2.3b}$$

Equations (2.3*a*) can be expanded into the following series:

$$\left.\begin{aligned} v &= h_x-\epsilon[J(h,h_y)+\alpha y h_x+\nu h_y+F^{(x)}]+O(\epsilon^2), \\ u &= -h_y-\epsilon[J(h,h_x)-\alpha y h_y+\nu h_x-F^{(y)}]+O(\epsilon^2). \end{aligned}\right\} \tag{2.4}$$

Substituting (2.4) into (2.3*b*) and omitting small terms, we get

$$h_t-\nabla\cdot[hJ(h,\nabla h)]-\alpha h h_x = \nu\nabla\cdot(h\nabla h)+(F^{(y)}h)_x-(F^{(x)}h)_y, \tag{2.5}$$

where $J(h,\nabla h) = h_x\nabla h_y-h_y\nabla h_x$ is the Jacobian operator. This equation was derived by Williams & Yamagata (1986).

Consider a steady flow:

$$h = H(y), \tag{2.6a}$$

where

$$H_y = 0 \quad \text{for } y < y_- \quad \text{or } y > y_+, \tag{2.6b}$$

and the boundaries $y_\pm$ of the flow may be, in principle, equal to $\pm\infty$. Evidently, flow (2.6) must be supported by the external forcing:

$$F^{(y)} = 0, \quad F^{(x)} = \nu H_y. \tag{2.7}$$

In order to examine the stability of (2.6), we assume that

$$h(x,y,t) = H(y) + h'(x,y,t) \quad (|h'| \ll H) \tag{2.8}$$

and then substitute (2.7)–(2.8) into (2.5). Omitting nonlinear terms, we obtain:

$$h'_t - (HH_{yy}h'_x)_y + \nabla\cdot(HH_y\nabla h'_x) - \alpha H h'_x = \nu\nabla(H\nabla h'). \tag{2.9}$$

We seek a solution in the form of a harmonic disturbance:

$$h'(x,y,t) = \operatorname{Re}\{\phi(y)\exp[ik(x-ct)]\}. \tag{2.10}$$

Substitution of (2.10) into (2.9) yields:

$$c\phi + [H(H_{yy}\phi - H_y\phi_y)]_y + k^2HH_y\phi + \alpha H\phi = -i\frac{\nu}{k}[(H\phi_y)_y - k^2H\phi]. \tag{2.11a}$$

We assume that the disturbance is localized near the flow:

$$\phi(y_\pm) = 0. \tag{2.11b}$$

If the eigenvalue of the boundary-value problem (2.11) has a negative imaginary part, flow (2.6) is unstable.

2.2. *Stability theorem for inviscid flows with monotonic profile*

Introducing

$$\hat{\phi} = \phi|_{\nu=0},$$

we substitute $\nu = 0$ into (2.11):

$$c\hat{\phi} + [H(H_{yy}\hat{\phi} - H_y\hat{\phi}_y)]_y + k^2HH_y\hat{\phi} + \alpha H\hat{\phi} = 0. \tag{2.12a}$$

$$\hat{\phi}(y_\pm) = 0. \tag{2.12b}$$

In order to prove the stability of flows with monotonic profile, we shall rewrite (2.12) in terms of a new variable

$$\psi = \frac{1}{H_y}\hat{\phi} \tag{2.13}$$

(which represents the displacement of the interface). Substitution of (2.13) into (2.12) yields

$$H_y(c+\alpha H)\psi - [H(H_y)^2\psi_y)]_y + k^2H(H_y)^2\psi = 0, \tag{2.14a}$$

$$\psi < \infty \quad \text{as } y \to y_\pm. \tag{2.14b}$$

Next we multiply (2.14*a*) by ψ^*, where the asterisk denotes complex conjugate, and integrate it with respect to y over $(-\infty,\infty)$. Integrating by parts and using the boundary conditions (2.14*b*), we obtain

$$cI_1 + I_2 = 0; \tag{2.15}$$

where $I_1 = \int_{y_-}^{y_+} H_y|\psi|^2\,dy, \quad I_2 = \int_{y_-}^{y_+} [H(H_y)^2(|\psi_y|^2 + k^2|\psi|^2) + \alpha HH_y|\psi|^2]\,dy.$

Both I_1 and I_2 are real quantities and, if

$$I_1 \neq 0, \tag{2.16}$$

then $c = -I_2/I_1$ is also real, which corresponds to neutral stability. The validity of (2.16) can be guaranteed by one of the following conditions:

$$H_y(y) \text{ does not change sign,} \tag{2.17}$$

or

$$\alpha = 0. \tag{2.18}$$

(2.17) makes the integrand of I_1 sign-definite, while (2.18) implies

$$I_2 \neq 0,$$

which, together with (2.15), seems also to guarantee (2.16).

However, the stability condition (2.18) contradicts the result by Pavia (1992), who found (numerically) an example of an unstable flow with $\alpha = 0$ and a non-monotonic profile. The contradiction can be resolved if we observe that the change of variables (2.13) in the non-monotonic case may be singular:

$$\psi \approx \frac{\text{const}}{y-y_*} \rightarrow \infty \quad \text{as } y \rightarrow y_*,$$

where y_* is the point where $H(y)$ has an extremum: $H_y(y_*) = 0$. As a result, both I_1 and I_2 may diverge.

Thus, the stability can be guaranteed only by condition (2.17), while the stability properties of flows with non-monotonic profile are, at this stage, unknown.

2.3. *The singularity at $H_y = 0$*

It should be emphasized, that the above singularity cannot be eliminated by the inverse change of variables $\psi \rightarrow \hat{\phi}$, as the original boundary-value problem (2.12) is also singular for non-monotonic flows. Indeed, the coefficient of the highest derivative in (2.12*a*) is proportional to H_y and therefore vanishes at $y = y_*$. As a result, $\hat{\phi}$ is not a well-behaved function:

$$\hat{\phi} \sim \text{const}_1(y-y_*) + \text{const}_2\,[1+\kappa(y-y_*)\ln(y-y_*)] \quad \text{as } y \rightarrow y_*, \tag{2.19}$$

where κ is a constant which depends on the Taylor expansion of $H(y)$ at $y = y_*$ (expansion (2.19) can be obtained using the Frobenius method). Furthermore, as the logarithm of a sign-indefinite argument is a multiple-valued function, it is unclear which branch should be chosen when $y-y_*$ changes sign. In other words, we need an additional condition matching $\hat{\phi}(y_*+0)$ to $\hat{\phi}(y_*-0)$. (It is worth noting that, although H_y also vanishes at $y = y_\pm$, the boundary conditions (2.12*b*) eliminate the logarithm from (2.19), and $\hat{\phi}$ is a single-valued function there.)

Similar logarithmic singularities also occur in the classical problem of critical levels (e.g. Dikiy 1976). However, in contrast to (2.12*a*), the critical-level equation is singular only for real c, and the singularity can be regularized by the assumption that c has a small imaginary part, modelling a weak viscosity). The solution with $\operatorname{Im} c < 0$ is unique and in the limit $\operatorname{Im} c \rightarrow 0$ establishes the branch of the logarithm that should be chosen in (2.19) when $y-y_*$ changes sign. It turns out that this approach does not work for our boundary-value problem (2.12), as the coefficient of the highest derivative in (2.12*a*) does not depend on c and therefore remains zero for $\operatorname{Im} c \neq 0$.

In order to regularize the singularity at $H_y = 0$, we should return to the viscous equations (2.11). In contrast to (2.12*a*), the coefficient of the highest derivative of (2.11*a*)

$$H(H_y - \mathrm{i}\,\nu/k)$$

does not vanish anywhere in $[y_-, y_+]$, which means that the solution is regular. Having calculated ϕ in the vicinity of the point $y = y_*$ (where $H_y = 0$), we shall take the limit $\nu \to 0$ and obtain $\hat{\phi}$.

Next we expand equation (2.11 *a*) about y_* and, assuming that $1 \gg \nu \gtrsim O(|y-y_*|)$, omit small terms:

$$(c+\alpha H_* + H_* H_*''')\phi - H_*\left[H_*''(y-y_*) - \mathrm{i}\frac{\nu}{k}\right]\phi_{yy} = 0,$$

where $$H_* = H(y_*), \quad H_*'' = H_{yy}(y_*), \quad H_*''' = H_{yyy}(y_*).$$

The general solution to this equation is

$$\phi = c_1[z+O(z^2)] + c_2[1+\kappa z \ln z + O(z^2)],$$

where $$z = y - y_* - \mathrm{i}\frac{\nu}{H_*'' k},$$

$c_{1,2}$ are constants, and

$$\kappa = c + \alpha H_* + H_* H_*''' \tag{2.20}$$

(here and hereinafter we assume that $H_*'' \neq 0$). Bearing in mind that $\nu > 0$, we take the limit $\nu \to 0$ and use the formula $\ln z = \ln|z| + \mathrm{i}\arg z$. In terms of $\hat{\phi} = \lim_{\nu\to+0}\phi$ we obtain

$$\hat{\phi} = \begin{cases} c_1(y-y_*) + c_2\{1+\kappa(y-y_*)[\ln|y-y_*| + \mathrm{i}\pi]\} + O[(y-y_*)^2] \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } y < y_*, \\ c_1(y-y_*) + c_2\{1+\kappa(y-y_*)[\ln|y-y_*| \\ \quad + \mathrm{i}\pi(1+\operatorname{sign} H_*'' k)]\} + O[(y-y_*)^2] \quad \text{for } y > y_*. \end{cases} \tag{2.21}$$

Using asymptotics (2.21), we can match the numerical solution of the inviscid boundary-value problem (2.12) across the singularity at $y = y_*$. We can also derive from (2.21) the formal matching conditions

$$\left.\begin{aligned} &\hat{\phi}(y_*+0) = \hat{\phi}(y_*-0), \\ &\hat{\phi}_y(y_*+0) = \hat{\phi}_y(y_*-0) + \frac{\mathrm{i}\pi\kappa}{H_*|H_*''|\operatorname{sign} k}\hat{\phi}(y_*), \end{aligned}\right\} \tag{2.22}$$

which supplement (2.12).

2.4. *Analytical results*

In this subsection we shall answer the following two questions:

(i) Is it possible that the solution of the inviscid regularized boundary-value problem (2.12, 2.22) has a real dispersion relation $c(k)$ corresponding to neutral stability?

(ii) Is the singularity crucial for the existence of instability?

It turns out that the answers to both questions are 'no'.

(i) If $\operatorname{Im} c = 0$, $\hat{\phi}$ and $\hat{\phi}^*$ both satisfy (2.12 *a*). It is easy to verify that their Wronskian must be proportional to $H^{-1}(y)$ everywhere except, possibly, for the singular point $y = y_*$:

$$W(\hat{\phi}, \hat{\phi}^*) = \hat{\phi}_y \hat{\phi}^* - \hat{\phi}\hat{\phi}_y^* = \begin{cases} \text{const}_+ H^{-1} & \text{for } y > y_*, \\ \text{const}_- H^{-1} & \text{for } y > y_*. \end{cases}$$

Substituting the boundary conditions (2.12 *b*) into $W(\hat{\phi}, \hat{\phi}^*)$, we see that $\text{const}_+ = \text{const}_- = 0$ and

$$W(\hat{\phi}, \hat{\phi}^*) \equiv 0 \quad \text{for } y \in [y_-, y_+]. \tag{2.23}$$

On the other hand, the matching conditions (2.22) entail

$$W(y_*+0) = W(y_*-0) + \frac{\mathrm{i}\pi\kappa}{H_*|H''_*|\,\mathrm{sign}\,k}|\hat{\phi}(y_*)|^2,$$

which is compatible with (2.23) only if

$$\kappa = 0 \tag{2.24}$$

or

$$\hat{\phi}(y_*) = 0. \tag{2.25a}$$

Substituting (2.20) into (2.24), we obtain

$$c = -\alpha H(y_*) - H(y_*)\,H_{yyy}(y_*). \tag{2.25b}$$

Either condition (2.25) represents an additional constraint imposed on the solution (eigenfunction or eigenvalue) of the boundary-value problem (2.12, 2.22) and, obviously, cannot hold for all values of k. In other words, $c(k)$ may be real only for isolated values of the wavenumber.

This conclusion is confirmed by the results of numerical integration of (2.12, 2.21) (see figure 2) for the flow described by

$$H(y) = 1-y^2, \tag{2.26}$$

$$y_\pm = \pm 1. \tag{2.27}$$

It should be noted that the profile of this flow is not smooth at $y_\pm$ (see figure 1) and therefore the boundary conditions (2.12b) should be generalized:

$$\frac{\hat{\phi}}{H_y} < \infty \quad \text{as } y \rightarrow y_\pm \tag{2.28}$$

(see Appendix A).

(ii) Consider the boundary-value problem (2.12, 2.22) in the limit $\alpha, k^2 \rightarrow 0$, in which case we can expand the solution in powers of these small parameters (a similar perturbation method was used by Griffiths, Killworth & Stern (1982)). We shall also assume that the global upper-layer depth difference is small:

$$\Delta H = H(y_+) - H(y_-) \ll 1. \tag{2.29}$$

The asymptotic solution of the stability boundary-value problem in this case is considered in Appendix B and yields the following dispersion relation:

$$k^2 \int_{y_-}^{y_+} H(H_y)^2\,\mathrm{d}y + c\,\Delta H + c^2 \int_{y_-}^{y_+} H_y F(y)\,\mathrm{d}y + \alpha^2 \int_{y_-}^{y_+} HH_y\,G(y)\,\mathrm{d}y$$
$$+ c\alpha \int_{y_-}^{y_+} H_y(G+HF)\,\mathrm{d}y = 0, \tag{2.30}$$

where

$$F(y) = \int \frac{1}{H(H_y)^2}(H-\bar{H})\,\mathrm{d}y, \quad G(y) = \int \frac{1}{H(H_y)^2}(H^2-\bar{H}^2)\,\mathrm{d}y,$$

$$\bar{H} = \tfrac{1}{2}[H(y_+)+H(y_-)].$$

We shall assume for simplicity that $H(y)$ has only one extremum – maximum at $y = y_*$:

$$H_y \gtrless 0 \quad \text{for } y \lessgtr y_*, \quad H_{yy}(y_*) < 0,$$

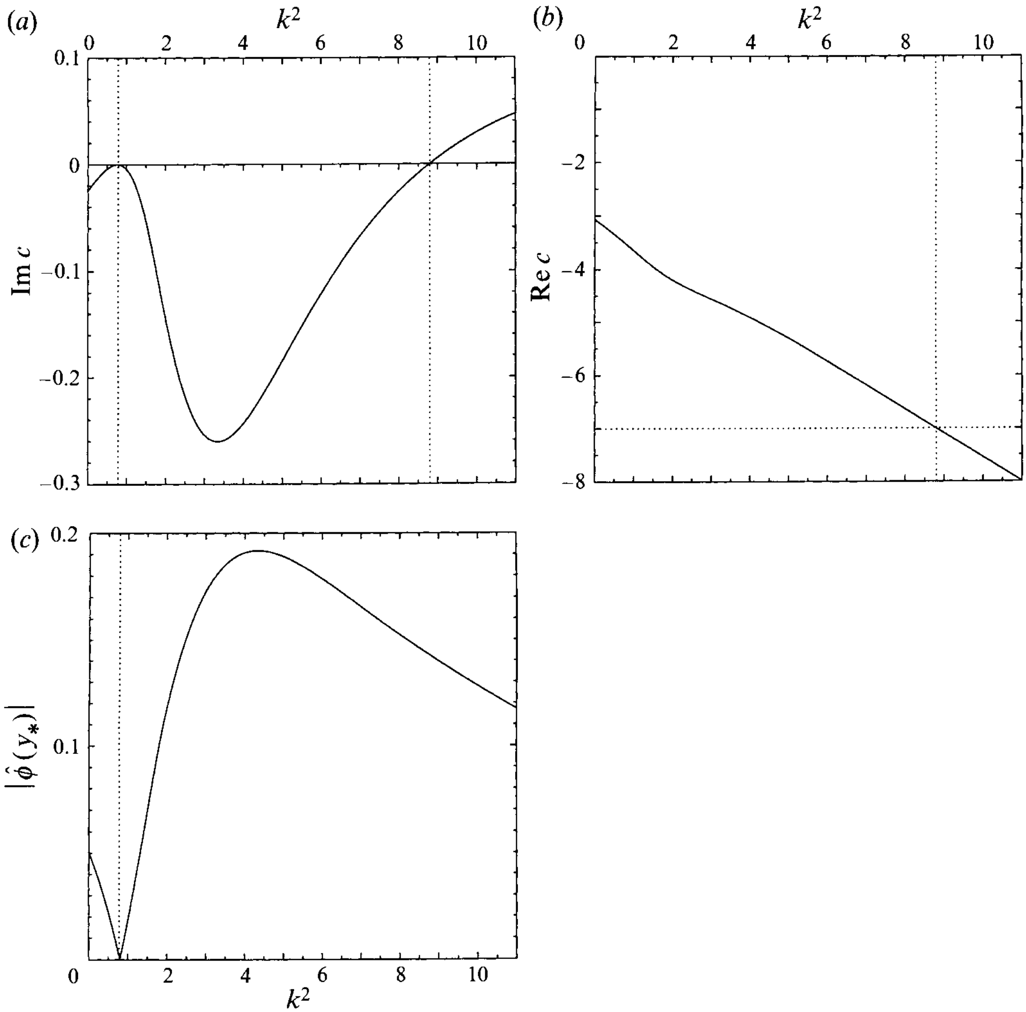

FIGURE 2. Dispersion relation of the boundary-value problem (2.12, 2.22) ('very thin' upper layer). $\alpha = 7$, $H(y) = 1-y^2$, $y \in [-1, 1]$. (*a*, *b*) Phase velocity c *vs.* wavenumber k. (*c*) Absolute value of the eigenfunction at the singular point *vs.* k. Observe that $\operatorname{Im} c$ vanishes only where $|\hat{\phi}(y_*)| = 0$, or where $\operatorname{Re} c = \alpha H_*$.

in which case it is easy to prove that the constant of integration in $F(y)$ can be chosen such that

$$F \gtrless 0 \quad \text{for } y \lessgtr y_*.$$

Hence, the coefficient of c^2 in the quadratic equation (2.30) is positive and, since the coefficient of k^2 is also positive, the roots of (2.30) are complex for sufficiently large k^2 (instability).

Remarkably, there was no need to use the regularizing conditions (2.22) in the derivation of (2.30), as all integrals in (2.30) converge at $y = y_*$ (see the last paragraph of Appendix B). Physically, this means that the contribution of the singular point is negligible. This, of course, applies only to the case where ΔH, α, $k^2 \ll 1$. However, the conclusion that the singularity is not crucial for the instability seems interesting and deserves attention (it also coincides with a similar conclusion by Griffiths *et al.* (1982) derived for ageostrophic flows with $\Delta H = \alpha = 0$).

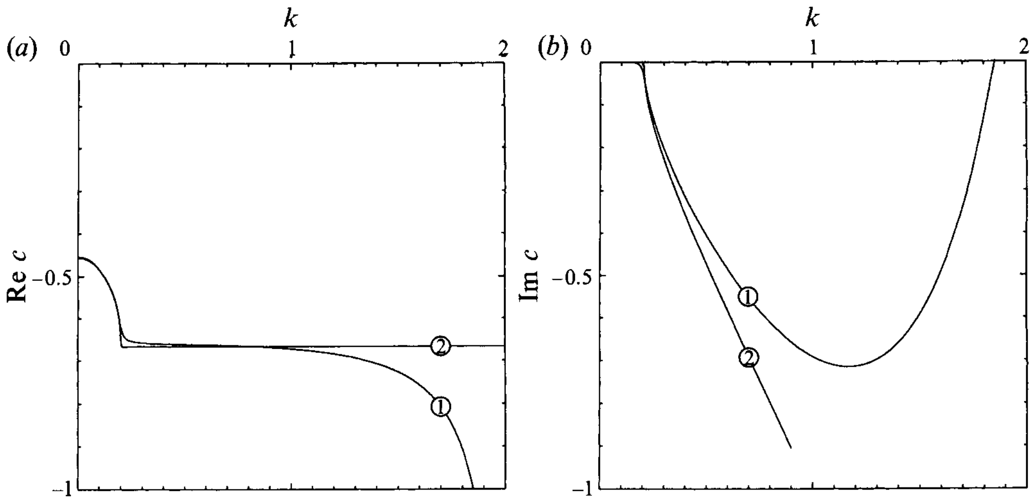

FIGURE 3. Dispersion relation of the boundary-value problem (2.12, 2.22) ('very thin' upper layer). $\alpha = 1$, $H(y) = 1-y^2$, $y \in [-1, 1]$. Curves (1) and (2) represent the numerical solution and the asymptotic solution (2.30), respectively.

The asymptotic dispersion relation (2.30) has been compared with numerical results for flow (2.26–2.27). Surprisingly, it satisfactorily described the exact solution up to and including $\alpha \approx 1$ (see figure 3).

Finally, if $H(y)$ has more than one extremum, the sign of the coefficient of c^2 in (3.30) is unclear and the instability cannot be proven. However, bearing in mind that each extremum, considered separately, would destabilize the flow, it is difficult to believe that their combination can make it stable.

2.5. *Numerical results*

In the beginning of the previous subsection, the solution to the stability boundary-value problem (2.12, 2.22) was proven to have no real eigenvalues. However, this does not prove instability, as the imaginary part of c may be positive and correspond to the decay of the disturbance and asymptotic stability. Although flows with $\Delta H, \alpha \ll 1$ were shown to have eigenvalues with $\operatorname{Im} c$ of either sign, the assumption of the instability for the general case needs to be verified numerically.

The boundary-value problem (2.12, 2.22) was integrated numerically for the flow profile (2.26) bounded by

$$y_- = -1, \quad y_+ \in (0, 1].$$

It was demonstrated that the maximum growth rate

$$\max_{k \in [0, \infty)} \{\operatorname{Im}[kc(k)]\}$$

grows with ΔH (see figure 4*a*) and even approaches infinity at $\Delta H = 1$. For negative values of y_+, the singular point disappears (see figure 1) and the unstable mode ceases to exist. (It should also be noted that, apart from the unstable mode, the boundary-value problem (2.5), (2.22) has an infinite number of stable modes which exist regardless of the singularity.)

The behaviour of spectral characteristics of the instability:

k_{max}, the wavenumber of maximum growth,

k_{mar}, the marginal wavenumber, such that $\operatorname{Im} c(k_{mar}) = 0$

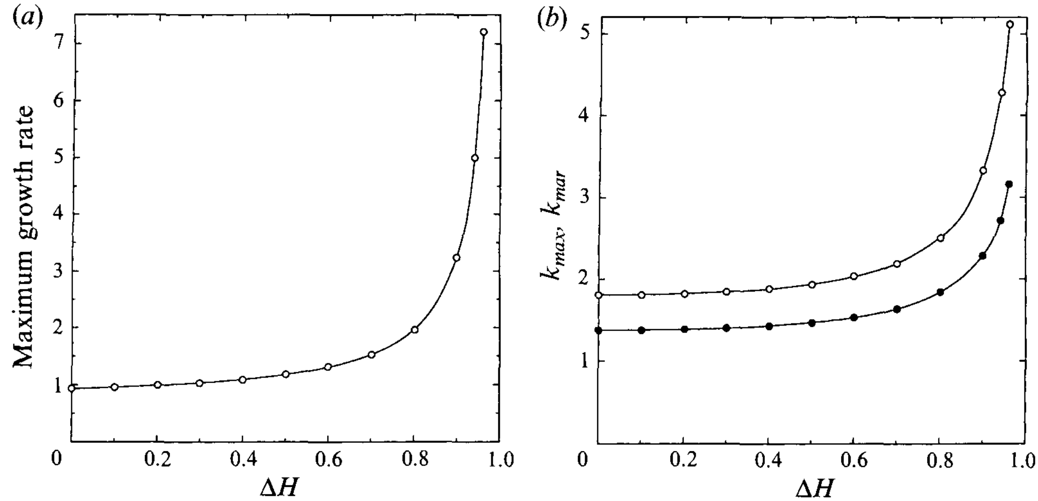

FIGURE 4. Dispersion relation of the boundary-value problem (2.12, 2.22) ('very thin' upper layer). $\alpha = 1$, $H(y) = 1-y^2$, $y \in [-1, y_+]$ (y varies from 0 to 1). (*a*) Maximum growth rate *vs.* global upper-layer depth difference ΔH. (*b*) Spectral characteristics of the instability: ○, the marginal wavenumber k_{mar} and ●, the wavenumber of maximum growth k_{max}, *vs.* ΔH.

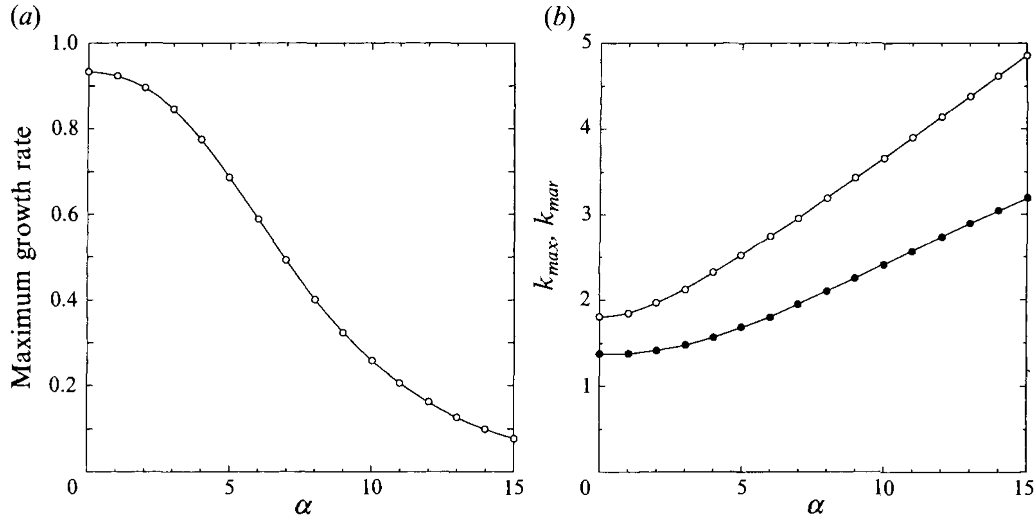

FIGURE 5. Dispersion relation of the boundary-value problem (2.12, 2.22) ('very thin' upper layer). $H(y) = 1-y^2$, $y \in [-1, 1]$. (*a*) Maximum growth rate *vs.* α. (b) Spectral characteristics of the instability: ○, the marginal wavenumber k_{mar} and ●, the wavenumber of maximum growth k_{max}, *vs.* α.

is shown in figure 4(*b*). Evidently, the instability of flows with $\Delta H \approx 1$ occurs for short wavelengths. We conclude that, if the singular point approaches one of the boundaries of the flow, the instability strengthens and shifts towards the short-wave region.

Figure 5 shows what happens if we fix ΔH and increase α: the maximum growth rate decreases, but k_{mar} and k_{max} grow. Hence, the β-effect weakens the instability, but is still unable to stabilize the flow.

We also examined the stability of the following family of flows with non-monotonic profiles:

$$H(y) = 1-y^2-\lambda y^4 \quad (\lambda \geqslant -\tfrac{1}{4});$$

which generalize (2.26). The whole family proved to be unstable, the instability being stronger for $\lambda < 0$ (wider flows) and weaker for $\lambda > 0$ (narrow flows). Together with all other evidence, this suggests that the singularity at $H_y = 0$ destabilizes all flows regardless of their profiles.

2.6. *Ageostrophic corrections*

In this subsection, we shall demonstrate that the introduction of infinitesimal viscosity is not the only way to regularize singularity at $y = y_*$: it can also be regularized by infinitesimal ageostrophic corrections. Our main motivation is the comparison of the results obtained via the two regularizations; surprisingly, they coincide only for the unstable wavenumbers $k < k_{mar}$. It also turns out that the new regularization links the problem at hand to the classical problem of critical levels.

Consider equations (2.3*a*) with $\nu = 0$, $F^{(x,y)} = 0$ and expand them up to the terms $O(\epsilon^2)$:

$$\left.\begin{aligned} v &= h_x - \epsilon[J(h, h_y) + \alpha y h_x] - \epsilon^2[h_{yt} + \ldots] + O(\epsilon^3) \\ u &= -h_y - \epsilon[J(h, h_y) - \alpha y h_y] - \epsilon^2[h_{xt} + \ldots] + O(\epsilon^3) \end{aligned}\right\} \tag{2.31}$$

(compare (2.31) with (2.4)). Then we substitute (2.31) into (2.3*b*):

$$h_t - \nabla \cdot [hJ(h, \nabla h)] - \alpha h h_x = \epsilon[\nabla \cdot (h \nabla h_t) + \ldots]. \tag{2.32}$$

Linearizing (2.32) against the background of the steady flow and substituting the harmonic-wave solution into the linearized equation, we obtain

$$c\phi + [H(H_{yy}\phi - H_y \phi_y)]_y + k^2 H H_y \phi + \alpha H \phi = \epsilon\{c[(H\phi_y)_y - k^2 H \phi] + \ldots\} \tag{2.33}$$

(compare (2.33) to (2.11*a*)). It should be noted that none of the terms concealed by the ellipsis '...' contains the second or higher derivatives; hence, the coefficient of the highest derivative in (2.33) is

$$H_y + \epsilon c,$$

which indicates that the singularity occurs when the velocity of the flow $u = -H_y$ matches the (scaled) phase speed of the disturbance, i.e. at the critical level. Now, if we take the limit $\epsilon \to 0$, the singularity shifts to the point where $H_y = 0$. In other words, our singularity is the limiting case of the critical-level singularity when $Ro \to 0$.

Another important observation is that, if $\operatorname{Im} c \neq 0$, the above coefficient of the highest derivative does not vanish, and therefore the ageostrophic correction regularizes equation (2.12*a*). In order to compare the ageostrophic regularization (2.33) with the viscous regularization (2.22), we observe that $-\mathrm{i}(\nu/k)$ can be replaced by ϵc only if $\operatorname{Im} c < 0$. This means that the two regularizations yield the same result only for unstable disturbances. If a disturbance is stable within the framework of viscous regularization ($\operatorname{Im} c > 0$), the corresponding ageostrophically regularized eigenvalue problem does not have any solution at all. In numerical calculations this manifests itself as follows: if we assume that the regularizing factor (ϵc) has a positive imaginary part, the solution to the regularized boundary-value problem yields $\operatorname{Im} c < 0$, and vice versa.

3. The case of 'thin' upper layer: $\delta \sim Ro$

This regime is described by the following set of equations (e.g. Benilov & Cushman-Roisin 1994):

$$\begin{aligned} &h_t + J(p, h) - \nabla \cdot [hJ(h, \nabla h)] - \alpha h h_x = 0, \\ &\alpha p_x + \nabla \cdot [hJ(h, \nabla h)] + \alpha h h_x = 0; \end{aligned} \tag{3.1}$$

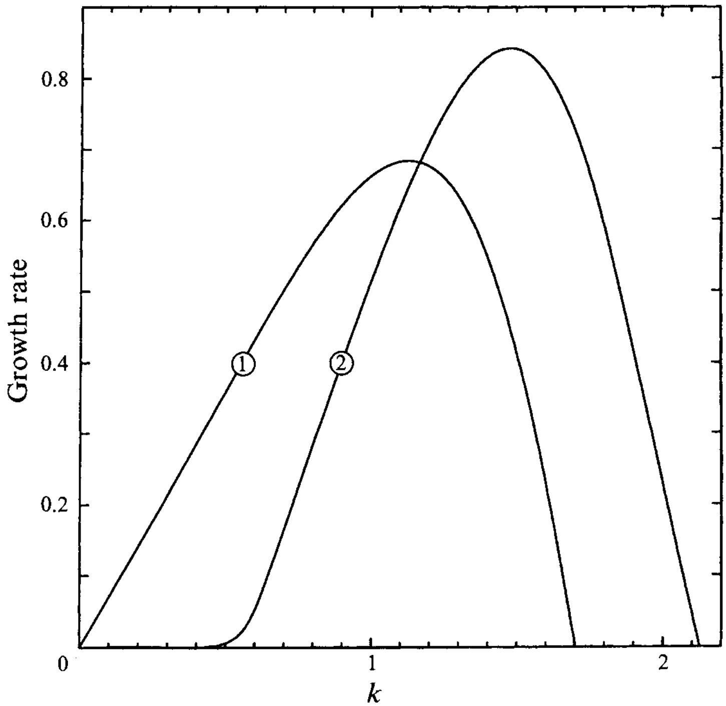

FIGURE 6. Comparison of the growth rates of instability for (1) the case of 'thin' upper layer; (2) the case of 'very thin' upper layer.

where p is the pressure in the bottom layer. Linearizing (3.1) against the background of the steady flow

$$h = H(y), \quad p = P(y)$$

and substituting the harmonic-wave solution into the linearized equations, we get

$$\frac{\alpha(c+P_y)}{\alpha+H_y}\hat{\phi}+[H(H_{yy}\hat{\phi}-H_y\hat{\phi}_y)]_y+k^2HH_y\hat{\phi}+\alpha H\hat{\phi} = 0. \tag{3.2}$$

In contrast to the 'very-thin-upper-layer' equation (2.12a), (3.2) has two singular points: at $H_y = 0$ and at $H_y = -\alpha$. It can be demonstrated (Benilov & Cushman-Roisin 1994) that all flows that satisfy one of the following conditions

$$H_y(y) \geqslant 0, \quad -\alpha \leqslant H_y(y) \leqslant 0, \quad H_y(y) \leqslant -\alpha,$$

have no singular points and are stable.

In what follows, we shall examine the stability of flows *with* singular points.

3.1. *The singularity at* $H_y = 0$

This singular point is very similar to its analogue in the case of 'very thin' upper layer, and can be regularized by either viscous or ageostrophic corrections. Accordingly, the flow (2.26) is unstable in the whole range of the parameters ΔH and α. The only difference between the two cases is that the instability of thin-upper-layer flows takes place in the long-wave region (see figure 6). This is important, as long-wave disturbances are responsible for meandering of the mean flow.

3.2. *The singularity at $H_y = -\alpha$*

It can be easily verified that this singularity cannot be regularized by inclusion of viscosity in the upper layer (which does not change the denominator of the first term in equation (3.2)). The viscosity in the bottom layer could do the job, but this would be meaningless from the physical point of view, as the coefficient of turbulent friction below thermocline in the real ocean is virtually zero. The most realistic way to regularize the singularity at $H_y = -\alpha$ is to take into account the terms that describe barotropic Rossby waves, i.e. consider the following system:

$$\left.\begin{aligned} &h_t + J(p,h) - \nabla\cdot[hJ(h,\nabla h)] - \alpha h h_x = 0, \\ &\epsilon[\Delta p_t + J(p,\Delta p)] + \alpha p_x + \nabla\cdot[hJ(h,\nabla h)] + \alpha h h_x = 0; \end{aligned}\right\} \tag{3.3}$$

where the small parameter ϵ can be treated as either the Rossby number or the non-dimensional depth of the upper layer (see Cushman-Roisin, Sutyrin & Tang 1992). The standard procedure of linearization and substitution of the harmonic-wave solution yields

$$(c+P_y)\phi - H_y\zeta + [H(H_{yy}\phi - H_y\phi_y)]_y + k^2 H H_y\phi + \alpha H\phi = 0, \tag{3.4a}$$

$$(c+P_y)\phi - (\alpha+H_y)\zeta + \epsilon[(c+P_y)(\zeta_{yy} - k^2\zeta) - P_{yyy}\zeta] = 0; \tag{3.4b}$$

where $\zeta(y)$ corresponds to the variable $p(x,y,t)$ (similar to ϕ corresponding to h). Unlike the previous case, the regularizing terms (terms $\sim\epsilon$) do not change the coefficient of the second derivative, but increase instead the order of the system of equations. We shall use the theorem (e.g. Wasow 1953) which reduces the fourth-order system with small coefficient of the fourth derivative, zero coefficient of the third derivative and sign-indefinite coefficient of the second derivative, to the second-order system with a regularized coefficient of the second derivative.

According to this theorem (see Appendix C), we can omit from (3.4*b*) all terms $\sim\epsilon$ and, at the same time, modify the coefficient of ζ as follows:

$$(c+P_y)\hat{\phi} - (\alpha + H_y + 0c)\hat{\zeta} = 0, \tag{3.5}$$

where the cap ^ indicates the limit $\epsilon\to 0$, and $+0c$ means that we should replace this term by $+\mu c$ ($\mu > 0$), solve equation (3.5) and then take the limit $\mu\to 0$. Then, substituting (3.5) into (3.4*a*), we obtain the following regularized version of (3.2):

$$\frac{\alpha(c+P_y)}{\alpha+H_y+0c}\hat{\phi} + [H(H_{yy}\hat{\phi} - H_y\hat{\phi}_y)]_y + k^2 H H_y\hat{\phi} + \alpha H\hat{\phi} = 0. \tag{3.6a}$$

This equation should be supplemented by the usual boundary conditions:

$$\frac{\hat{\phi}}{H_y} < \infty \quad \text{as } y\to y_{\pm}. \tag{3.6b}$$

Using the Wronskian method, it is easy to prove that this boundary-value problem may not have real dispersion relation. As this analytical result does not guarantee the existence of the unstable complex eigenvalues, the boundary-value problem (3.6) was integrated numerically for the flow profile (2.26) with

$$y_- = -1, \quad y_+ = 0,$$

(this flow has a singular point at $H_y = -\alpha$). The results obtained (figure 7) indicate that

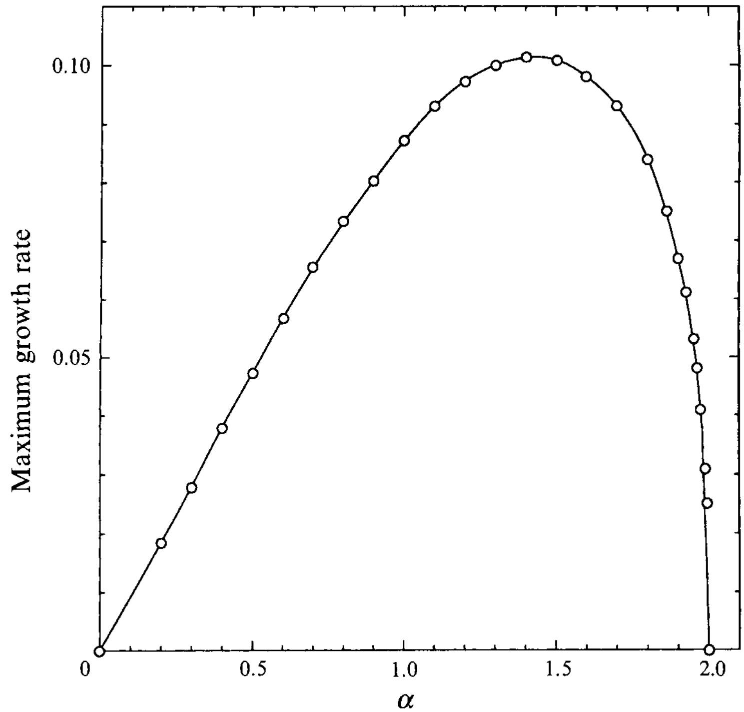

FIGURE 7. Maximum growth rate *vs.* α ('thin' upper layer). $H(y) = 1-y^2$, $y \in [-1, 1]$.

the solution exists for all values of α, that allow the existence of the singular point. It is worth noting that the asymmetry of the graph of the maximum growth rate versus α indicates that the growth rate depends on the velocity of the flow at the singular point. Indeed, if $\alpha \to 0$, the singular point moves closer to the right boundary of the flow, where the velocity ($\sim H_y$) is zero; as a result, the growth rate is smaller than that in the case $\alpha \to 2$ (where the singular point occurs at the velocity maximum).

4. The case of 'thick' upper layer: $\delta \sim 1$

As it turns out, the solution of the stability boundary-layer problem in this case has a solution only for finite values of regularizing factors (regardless of their physical meaning). The limiting problem has no solutions at all, which means neutral stability.

Taking into account the ageostrophic effects and viscosity in the upper layer, we can write the governing equations in the form

$$\left.\begin{aligned} &h_t + J(p,h) - \alpha h(1-h)\,h_x = \nabla\cdot[h(\epsilon h_t + \nu\nabla h)] + (F^{(y)}h)_x - (F^{(x)}h)_y,\\ &\alpha p_x + \nabla\cdot[h(1-h)\,J(h,\nabla h)] + \alpha h h_x = 0\end{aligned}\right\} \tag{4.1}$$

(system (4.1) with $\epsilon = \nu = F^{(y)} = F^{(x)} = 0$ was derived by Benilov (1992*a*)). The corresponding ordinary differential equation is

$$\alpha[c + P_y + \alpha H(1-H)]\,\phi \\ + H_y[H(1-H)(H_{yy}\phi - H_y\phi_y)]_y + k^2H(1-H)(H_y)^2\phi = \alpha\left[H\left(\epsilon c - \mathrm{i}\frac{\nu}{k}\right)\phi_y\right]_y. \tag{4.2}$$

4.1. *Singularity at $H_y = 0$*

Taking the limit $\epsilon, \nu \to 0$, we expand (4.2) about y_*:

$$\alpha[c-U+\alpha H(1-H)]\hat{\phi}-(H''_*)^2(y-y_*)^2\hat{\phi}_{yy}=0. \tag{4.3}$$

In contrast to the previous cases, the coefficient of the second derivative in (4.3) is proportional to $(y-y_*)^2$ (not to $(y-y_*)$), and the solution does not contain logarithms:

$$\hat{\phi}=c_1(y-y_*)^{\gamma_1}+c_2(y-y_*)^{\gamma_2}, \tag{4.4}$$

where $\gamma_{1,2}$ are the roots of the quadratic equation

$$\gamma(\gamma-1)=\frac{\alpha[c-U+\alpha H(1-H)]}{(H''_*)^2}$$

(compare (4.4) to (2.19)). This type of singularity does not allow derivation of any analytical estimate for the eigenvalue, and the problem was examined numerically.

4.2. *Numerical results*

First, equation (4.2) was integrated with small, but finite, values of ϵ and ν for the flow with profile (2.26–2.27). Surprisingly, the results obtained demonstrated that, in contrast to the previous cases, the solution strongly depends on the magnitudes of ϵ and ν and does not tend to a finite limit as ϵ, $\nu \to 0$.

In what follows, we shall consider the short-wave approximation ($k^2 \to \infty$) of (4.2) (which makes the results independent of the particular profile of the flow). In this case the eigenfunction is localized in the vicinity of the singular point (which was confirmed by the numerical results obtained for flow (2.26–2.27), and we can expand (4.2) as follows:

$$\phi_{yy}-\left[k^2+\frac{\tilde{c}}{(y-y_*)^2+\mu}\right]\phi=0, \tag{4.5a}$$

$$\mu=H_*\left[\frac{\epsilon\alpha}{H_*(1-H_*)(H''_*)^2}c-\mathrm{i}\frac{\nu}{k}\right],$$

$$\tilde{c}=\frac{\alpha}{H_*(1-H_*)(H''_*)^2}[c-U+\alpha H_*(1-H_*)];$$

where $\tilde{c}$ should be treated as the 'new' eigenvalue and μ is the regularizing factor. Equation (4.5a) is valid for any flow profile and short disturbance and, in this sense, is standard. As the solution is expected to be localized near the singular point, the boundaries of the flow can be shifted to infinity:

$$\phi(\pm\infty)=0. \tag{4.5b}$$

The boundary-value problem (4.5) was integrated numerically. Figure 8 shows the eigenvalue $\tilde{c}$ *vs.* the phase of the regularizing factor μ (as $\nu > 0$, it follows that $\arg\mu \in (-\pi, 0)$). Evidently, both real and imaginary parts of $\tilde{c}$ strongly depend on $\arg\mu$. It is worth noting, that sharp changes in $\mathrm{Re}\,\tilde{c}$ and $\mathrm{Im}\,\tilde{c}$ in the vicinity of $\arg\mu \to -\pi$ suggest the possibility of an asymptotic approach (the width of this boundary layer can be estimated as $O(^1/k^2)$). A similar pattern can be observed for the eigenfunction, whose structure in the vicinity of the singularity strongly depends on $\arg\mu$. Finally, it should be emphasized that $\mathrm{Im}\,\tilde{c}$ is positive for all values of $\arg\mu$, which agrees with the corresponding results for the thin-upper-layer regimes: short waves are stable.

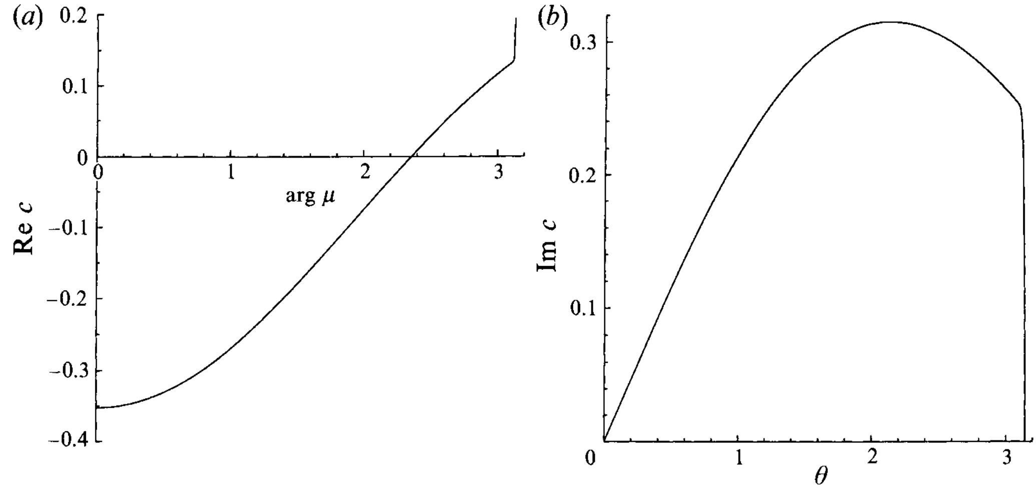

FIGURE 8. Real (*a*) and imaginary (*b*) parts of the eigenvalue of the boundary-value problem (4.5) ('thick' upper layer) *vs.* phase of the regularizing factor μ. $k^2 = 50$, $|\mu| = 0.002$.

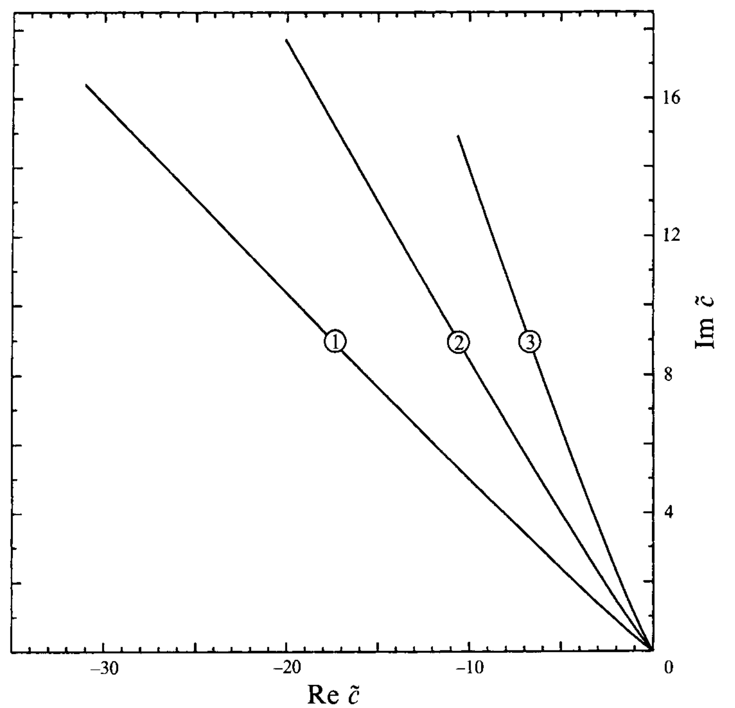

FIGURE 9. Eigenvalue of the boundary-value problem (4.5) ('thick' upper layer) *vs.* magnitude of the regularizing factor μ. (1) $k^2 = 100$, $\arg\mu = \frac{1}{6}\pi$; (2) $k^2 = 75$, $\arg\mu = \frac{1}{4}\pi$; (3) $k^2 = 50$, $\arg\mu = \frac{1}{3}\pi$.

Figure 9 demonstrates that $\tilde{c}$ tends to zero as $|\mu| \rightarrow 0$ regardless of $\arg\mu$ and the wavenumber k^2. This conclusion was confirmed for some higher-mode eigenvalues (boundary-value problem (4.5) describes an infinite number of modes).

Thus, all eigenvalues for all wavenumbers tend to zero, while the corresponding eigenfunctions do not have finite limits. Exactly the same pattern was observed in the classical problem of stability of the Poiseuille flow with the viscosity coefficient tending to zero (e.g. Dikiy 1976), where it was interpreted as the non-existence of a solution, i.e. stability. However, the Poiseuille flow can be (weakly) destabilized by finite viscosity. In other words, the stability of the inviscid Poiseuille flow is structurally unstable.

Following this example, we conclude that large-amplitude geostrophic flows with thick upper layer are stable, but, generally speaking, can be weakly destabilized by small viscosity ageostrophic effects or external forcing.

5. Conclusions

We have considered the stability of zonal geostrophic flows with large displacement of the interface. Attention was focused on zonal currents with non-monotonic profile $H(y)$ (where H is the depth of the upper layer). For such currents, the coefficient of the highest derivative in the corresponding stability boundary-value problem vanishes at $H_y = 0$. As a result, the eigenfunction is a multiple-valued function and needs regularization. Although this singularity is similar to the critical level, the simplistic introduction of infinitesimal viscosity through the assumption that the phase speed of the disturbance has a small imaginary correction, does not regularize the problem.

In order to regularize the boundary-value problem, one should derive the asymptotic equation which properly takes into account viscosity and then take the limit $\nu \rightarrow 0$ (where ν is the coefficient of viscosity).

(i) For flows with 'very thin' upper layer:

$$\frac{\text{depth of the upper layer}}{\text{total depth of the fluid}} \sim Ro^2,$$

we have proved the instability of all flows with small α and ΔH (where ΔH is the global upper-layer depth difference – see figure 1). It was also proved that the regularized boundary-value problem may not have a real dispersion relation (corresponding to neutral stability). Using these results together with the numerical results obtained for a special family of flow profiles:

$$H(y) = 1 - y^2 - \lambda y^4, \quad -1 \leqslant y \leqslant y_+, \quad y \in (0, 1], \tag{5.1}$$

(where $\lambda \geqslant -\frac{1}{4}$), we argue that the singular point destabilizes all flows with non-monotonic profiles.

It was also demonstrated that the singularity can be regularized by taking into account small ageostrophic corrections, in which case the coefficient of the highest derivative is

$$H_y - c\,Ro,$$

where c is the (scaled) phase speed of the perturbation. As H_y represents the velocity in the upper layer, this result links our singularity to the critical-level singularity. The regularized boundary-value problem can be obtained by taking the limit $Ro \rightarrow 0$ and the assumption $\operatorname{Im} c \neq 0$. Surprisingly, the results obtained via the two regularizations agree only for unstable disturbances: for the wavenumbers, where the viscosity-regularized equations have a stable solution, the ageostrophically regularized equations have no solution at all.

(ii) For flows with 'thin' upper layer:

$$\frac{\text{depth of the upper layer}}{\text{total depth of the fluid}} \sim Ro,$$

	H_y	α
Kuroshio	−0.027	0.021
Subarctic front	−0.024	0.015
Subtropical front	−0.025	0.044

TABLE 2. Ranges of slope of the interface and β-effect number in real oceans

we demonstrated that, apart from the singularity that occurs at $H_y = 0$, the stability boundary-value problem has a singularity at

$$H_y = -\alpha. \tag{5.2}$$

Accordingly, eastward flows can be unstable even in the case of a monotonic profile. The singularity at (5.2) can be regularized by taking into account barotropic Rossby waves (viscosity in the upper layer does not change the coefficient of the highest derivative in this case).

It is worth noting that the regime of thin upper layer is the most interesting from a physical point of view. It was demonstrated (Benilov & Reznik 1994) to include most of the oceanic frontal flows except the Gulf Stream (which is not geostrophic) and the Antarctic Circumpolar Current (which has a thick upper layer). Using estimates based on Roden's (1975) experimental (table 2), we conclude that the Kuroshio and subarctic front are unstable, while the subtropical front seems to be stable. However, Roden's (1975) data on the subtropical frontal flow indicate the existence of a weaker westward jet in between the two strong eastward jets, which correspond to at least two points where $H_y = 0$. Although each of the three jets is stable, these points should destabilize the subtropical frontal system as a whole.

(iii) For flows with thick upper layer:

$$\frac{\text{depth of the upper layer}}{\text{total depth of the fluid}} \sim 1,$$

the singularity occurs only at $H_y = 0$. In contrast to the previous cases, where the coefficient of the highest derivative was proportional to $(y-y_*)$, the coefficient of the highest derivative here is proportional to $(y-y_*)^2$, which changes the type of the singularity. Although the singularity can still be regularized by taking into account either viscosity in the upper layer or ageostrophic effects, the boundary-value problem has non-zero eigenvalues only for finite values of the regularizing factors (the coefficient of viscosity ν and Rossby number Ro). It was verified for short disturbances (and arbitrary flow profile) that in the limit ν, $Ro \rightarrow 0$ all eigenvalues for all wavenumbers tend to zero, while the eigenfunction do not have finite limits at all, which corresponds to neutral stability. It can be conjectured, however, that flows with thick upper layer may be weakly destabilized by weak friction, ageostrophic effects of forcing.

Appendix A. Boundary conditions for flows with non-smooth profiles

If $H(y)$ is not smooth at $y = y_{\pm}$:

$$H_y = H'_{\pm} \neq 0 \quad \text{as}\, y \rightarrow y_{\pm} \mp 0,$$

the boundary condition (2.12*b*) should be modified. First, we 'imagine' that in the vicinity of y_+ and y_- there are narrow transitional intervals of width Δ, where H_y smoothly changes from $H'_\pm$ to zero. Now we can 'shift' (2.12 *b*) to the outer boundaries of these intervals:

$$\hat{\phi} = 0 \quad \text{at } y = y_\pm \pm \Delta. \tag{A 1}$$

As Δ is small, $\hat{\phi}$ is a fast-varying function, and we can omit non-derivative terms from (2.12 *a*):

$$[H(H_{yy}\hat{\phi} - H_y \hat{\phi}_y)]_y = 0. \tag{A 2}$$

Integrating (A 2) and taking into account (A 1) and the condition $H_y \equiv 0$ valid outside $(y_- - \Delta, y_+ + \Delta)$, we obtain

$$\hat{\phi} = H_y(y) \quad \text{for } y \in [y_- - \Delta, y_-), \quad [y_+, y_+ + \Delta].$$

Now we take the limit $\Delta \to 0$ and obtain condition (2.28).

Appendix B. Perturbation theory for k^2, α, $\Delta H \to 0$ (very thin upper layer)

It is convenient to introduce

$$\chi = H(H_{yy}\hat{\phi} - H_y \hat{\phi}_y), \tag{B 1}$$

and rewrite (2.12 *a*) in the form

$$\left(\frac{1}{H_y}\hat{\phi}\right)_y = \frac{1}{H(H_y)^2}\chi, \quad \chi_y = (k^2 H H_y + c + \alpha H)\hat{\phi}. \tag{B 2a, b}$$

Substitution of (2.12 *b*) or (2.28) into (B 1) yields the boundary condition which holds for both smooth and non-smooth (at $y = y_\pm$) flows:

$$\chi = 0, \quad \text{at } y = y_\pm. \tag{B 3}$$

It is convenient to split $H(y)$ as follows

$$H(y) = \eta(y) + \xi(y),$$

where

$$\eta(y_+) = \eta(y_-) = \bar{H},$$

$$\xi(y_+) = \xi(y_-) + \Delta H.$$

Condition (2.29) entails

$$\eta \sim 1, \quad \xi \sim \Delta H \ll 1.$$

Now, assuming that

$$\Delta H \sim k \sim \alpha \ll 1,$$

we expand the solution to (B 2–B 3) as follows:

$$\hat{\phi} = \hat{\phi}^{(0)} + \hat{\phi}^{(1)} + \dots, \quad \chi = \chi^{(1)} + \dots, \quad c = c^{(1)} + \dots;$$

and obtain

$$\left(\frac{1}{\eta_y}\hat{\phi}^{(0)}\right)_y = 0; \tag{B 4}$$

$$\chi_y^{(1)} = (c^{(0)} + \alpha\eta)\hat{\phi}^{(0)}, \quad \chi^{(1)}(y_\pm) = 0; \tag{B 5}$$

$$\left(\frac{1}{\eta_y}\hat{\phi}^{(1)} + \frac{1}{(\eta_y)^2}\xi_y \hat{\phi}^{(0)}\right)_y = \frac{1}{\eta(\eta_y)^2}\chi^{(1)}; \tag{B 6}$$

$$\chi_y^{(2)} = (c^{(0)} + \alpha\eta)\hat{\phi}^{(1)} + (c^{(1)} + \alpha\xi)\hat{\phi}^{(0)}, \quad \chi^{(2)}(y_\pm) = 0 \tag{B 7a, b}$$

Equations (B 4–B 6) can be solved easily:

$$\hat{\phi}^{(0)} = \eta_y,$$
$$\chi^{(1)} = c^{(1)}(\eta-\bar{H})+\tfrac{1}{2}(\eta^2-\bar{H}^2),$$
$$\hat{\phi}^{(1)} = \xi_y+\eta_y\int\frac{1}{\eta(\eta_y)^2}\chi^{(1)}\,\mathrm{d}y.$$

Then, integrating (B 7*a*) and taking into account (B 7*b*), we eliminate $\chi^{(2)}$:

$$\int_{y_-}^{y_+}(c^{(0)}+\alpha\eta)\,\hat{\phi}^{(1)}+(c^{(1)}+\alpha\xi)\,\hat{\phi}^{(0)}\,\mathrm{d}y = 0$$

and substitute $\hat{\phi}^{(0)}$ and $\hat{\phi}^{(1)}$. Omitting the superscripts and replacing η by $H\,(\eta\approx H)$, we obtain (2.30).

It is worth noting that we did not need to use the regularizing conditions (2.22) in the derivation of (2.30), because the zeroth-order eigenfunction vanishes at the critical point, that is $\hat{\phi}^{(0)}(y_*)=0$, and the singularity cancels out.

Appendix C. Reduction of (3.4*b*) to (3.5)

Obviously, small terms $\sim\epsilon$ can be neglected for all y except $y\rightarrow y_*$ (where y_* is the singular point as before). In order to clarify the structure of the solution in the vicinity of y_*, we shall expand $H(y)$ about y_*:

$$H = H_*-\alpha(y-y_*)+\tfrac{1}{2}H''_*(y-y_*)^2+\ldots.$$

We assume that in the vicinity of ϕ and ζ are fast-varying functions, such that

$$|\phi_{yy}|\gg|\phi_y|\gg|\phi|,\quad |\zeta_{yy}|\gg|\zeta_y|\gg|\zeta|.$$

Now we shall expand (3.4) about y_*:

$$\left.\begin{aligned}\zeta &= H_*\,\phi_{yy},\\ (c-U)\,\phi-H_*\,H''_*(y-y_*)\,\zeta+\epsilon(c-U)\,\zeta_{yy} &= 0;\end{aligned}\right\}\tag{C 1}$$

where $U=-P_y(y_*)$ is the velocity in the bottom layer. System (C 1) can be reduced to a single equation of the fourth order:

$$(c-U)\,\phi-H_*\,H''_*(y-y_*)\,\phi_{yy}+\epsilon(c-U)\,H_*\,\phi_{yyyy}=0.\tag{C 2}$$

The coefficient of the second derivative in the limiting equation

$$(c-U)\,\hat{\phi}-H_*\,H''_*(y-y_*)\,\hat{\phi}_{yy}=0,$$

vanishes at $y=y_*$ and its solution is a multiple-valued function. In the previous case, we regularized similar equations by inserting an infinitesimal imaginary correction into the coefficient of ϕ_{yy}:

$$(c-U)\,\hat{\phi}-H_*[H''_*(y-y_*)\pm\mathrm{i}0]\,\hat{\phi}_{yy}=0.\tag{C 3}$$

However, the solution to (C 3) depends on the sign of this infinitesimal correction, which is *a priori* unclear.

In order to determine the sign of the regularizing factor, we shall use the theorem proven by Wasow (1953). In application to (C 2), it states that in the limit $\epsilon\rightarrow 0$, two of the four linear independent solutions of (C 2) tend to the corresponding solutions of (C 3) provided the sign of i0 is such that

$$K(y) = \mathrm{Re}\int\left(\frac{H''_*(y-y'')\pm\mathrm{i}0}{c-U}\right)^{1/2}\mathrm{d}y$$

is a monotonic function.

In order to find the sign i0 that provides the 'monotonicity' of $K(y)$, we shall replace 0 by a finite constant μ (we shall take the limit $\mu \to 0$ afterwards) and evaluate the integral:

$$K(y) \sim \mathrm{Re}\frac{[H_*''(y-y_*)+\mathrm{i}\mu]^{\frac{3}{2}}}{H_*''(c-U)}.$$

Using the so-called Stokes lines, it is easy to understand that $K(y)$ is a monotonic function only if μ has the same sign as $\mathrm{Im}\, c$, and (C 3) should be rewritten as

$$(c-U)\hat{\phi}-H_*[H_*''(y-y_*)+0c]\,\hat{\phi}_{yy}=0.$$

In terms of $\hat{\zeta}=\lim_{\epsilon\to\infty}\zeta$, this equation can be rewritten as

$$\left.\begin{aligned}&\hat{\zeta}=H_*\,\hat{\phi}_{yy},\\&(c-U)\,\hat{\phi}-H_*[H_*''(y-y_*)+0c]\,\hat{\zeta}=0.\end{aligned}\right\} \qquad (\mathrm{C}\ 4)$$

Finally, the 'non-expanded analogue' of (C 4) is system (3.5).

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